

## Préambule

La gestion financière responsable vise la maximisation de la richesse relative au risque dans le respect du bien commun des diverses parties prenantes, actuelles et futures, tant de l'entreprise que de l'économie en général. Bien que ce concept ne soit pas en contradiction avec la définition de la théorie financière moderne, les applications qui en découlent exigent un comportement à la fois financièrement et socialement responsable.

La gestion responsable des risques financiers, le cadre réglementaire et les mécanismes de saine gouvernance doivent pallier aux lacunes d'un système parfois trop permissif et naïf à l'égard des actions des intervenants de la libre entreprise.

Or, certaines pratiques de l'industrie de la finance et de dirigeants d'entreprises ont été sévèrement critiquées depuis le début des années 2000. De la bulle technologique (2000) jusqu'à la mise en lumière de crimes financiers [Enron (2001) et Worldcom (2002)], en passant par la mauvaise évaluation des titres toxiques lors de la crise des subprimes (2007), la fragilité du secteur financier américain (2008) et le lourd endettement de certains pays souverains, la dernière décennie a été marquée par plusieurs événements qui font ressortir plusieurs éléments inadéquats de la gestion financière. Une gestion de risque plus responsable, une meilleure compréhension des comportements des gestionnaires, des modèles d'évaluation plus performants et complets intégrant des critères extra-financiers, l'établissement d'un cadre réglementaire axé sur la pérennité du bien commun d'une société constituent autant de pistes de solution auxquels doivent s'intéresser tant les académiciens que les professionnels de l'industrie. C'est en mettant à contribution tant le savoir scientifique et pratique que nous pourrons faire passer la finance responsable d'un positionnement en périphérie de la finance fondamentale à une place plus centrale. Le développement des connaissances en finance responsable est au cœur de la mission et des intérêts de recherche des membres du Groupe de Recherche en Finance Appliquée (GReFA) de l'Université de Sherbrooke.

Les marchés Hors-Bourse ont reçu beaucoup d'attention depuis la crise financière de 2007-2009 et leur rôle central dans cette crise a généré de vives inquiétudes de sorte que plusieurs directives ont été émises par les organismes de régulations financières autour du monde pour en assurer le contrôle. Dans cet article, nous obtenons des solutions explicites aux systèmes d'équations différentielles non-linéaires qui décrivent l'évolution dans le temps des marchés Hors-Bourse du type de ceux étudiés par Darrell Duffie et ses multiples collaborateurs. Nous espérons que nos modèles théoriques pourront éclairer les participants et les régulateurs de ces marchés pour en assurer le meilleur fonctionnement.

# Some new results on Duffie-type OTC markets

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Abstract: The extended Wild sums considered in this article generalize the classical Wild sums of statistical physics. We first show how to obtain explicit solutions for the evolution equation of a large system where the interactions are given by a single, but general, interacting kernel which involves  $m$  components, for a fixed  $m \geq 2$ . We then show how to retain the explicit formulas for the case of OTC market models where the dynamics is more directly described by two (or more) kernels.

## 1 Introduction

After the publication of M. Kac's work (1956) [8], there was a renewed interest for the results of E. Wild (1951) [15]. This interest was mainly focused on the random matching of a large population of particles forming a diluted Maxwell gas. Here we develop an approach inspired by this body of work. To do so, we start with a sequence of dynamical sets of interacting components, one for each integer  $N$ . For these dynamical systems we can show that when  $N$  is large the probability is very small that a component has interacted more than once, directly or indirectly, up to time  $t$ , with any other component. Thanks to this fundamental property, we can link the microscopic and macroscopic levels using results from the theory of continuous-time Markov chains.

The Wild sum is a series construction which gives the solution of a given evolution equation in the statistical physics of gases as first appeared in the work of Wild [15]. Note that the classical expression of a Wild sum is described by binary trees. Inspired by these ideas, S. Tanaka [12] and H. Tanaka [11] defined an extension of Wild's sum for solving certain non-linear differential equations of spaces of measures, so the expression of this sum is described by appropriate trees. However, the problem of showing the existence of these sums remains wide open in general.

The recursive time relaxed Monte Carlo methods of Trazzi, Pareschi and Wennberg [14] are based on generalized Wild sums. However, the lack of explicit formulas for these sums constitutes a handicap for the efficiency of the above

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methods as well as others also based on extended Wild sums (see [13], for instance).

Carlen, Carvalho and Gabetta [4] obtain Wild sum formulas which are quite explicit for the solution of the Kac equation. Their binary trees are obtained, in the spirit of McKean, from commutator formulas for Lie algebras, leading them to groupings of interaction trees. Consequently, our more general interaction trees are different from theirs even in the binary case.

The aim of this paper is to propose a combinatorial formula for extended Wild sums which are solutions of certain evolution equations and more precisely in the context of interactions involving  $m$  components,  $m \geq 2$ .

In Section 3 of Bélanger and Giroux [1], the explicit formulas for the Wild sums were used to obtain the convergence of the solution of the evolution equation to a steady state. This is one of the important applications permitted by the tractability of our explicit formulas.

The article is organized as follows. In Section 2, we introduce the types of combinatorial trees which are going to be useful in the expression of the solution of the evolution equation in terms of interaction trees. In Section 3 and 4 we consider interactions involving  $m$  components, for  $m \geq 2$ , and we suppose that the intensities of these dynamics have an adequate dependence on  $N$ . Our techniques enable us to obtain an explicit formula for the solution of the associated system of differential equations. In Section 4, we show how to retain the explicit formulation of the solutions in the case of OTC market models described by two kernels.

## 2 Combinatorial trees

We assume that the reader is familiar with the basic definitions of trees. A rooted tree is a tree with a designated node called the root. A rooted tree in which the rooted node has one child is a planted tree. An  $m$ -ary tree is a rooted tree where each of its node is either a leaf (that is, it has no child) or it has exactly  $m$  children. The leaves are called external nodes and those nodes with  $m$  children, internal nodes. Note that we do not consider the root of the tree as an internal node.

An ordered tree is a rooted tree in which the children of each node are assigned a fixed ordering.

A rooted tree is called an  $(m, 1)$ -ary tree if each internal node has either one child or exactly  $m$  children. In this article, we will work with ordered  $m$ -ary trees and ordered  $(m, 1)$ -ary trees.

Let  $\mathbb{A}_n$  denote the set of  $m$ -ary ordered trees with  $n$  internal nodes. Each tree in  $\mathbb{A}_n$  has  $(m - 1)n + 1$  leaves and each tree can be obtained by adding an internal node on a leaf of a tree in  $\mathbb{A}_{n-1}$  (taking into account the order). Hence

the number of trees in  $\mathbb{A}_n$  is  $\#_m(n) = \prod_{k=1}^{n-1} ((m - 1)k + 1)$ .

### 3 The dynamics

Let  $N$  be the (large) number of interacting components. Let  $m$  ( $m \geq 2$ ) be the fixed number of components involved in each interaction. We suppose that all components take their values in a measurable space,  $(E, \mathcal{E})$ , (one can think of  $(\mathbb{R}^d, B(\mathbb{R}^d))$  or simply a finite set) and their interactions are given by a symmetric probability kernel  $Q$  on the product space  $(E^m, \mathcal{E}^{\otimes m})$ . That is, the function  $Q(x_1, x_2, \dots, x_m; C_1 \times \dots \times C_m)$ : is measurable in  $(x_1, x_2, \dots, x_m)$ ; is a probability measure in  $(C_1 \times \dots \times C_m)$ ; and satisfies  $Q(x_1, x_2, \dots, x_m; C_1 \times \dots \times C_m) = Q(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(m)}; C_{\sigma(1)} \times C_{\sigma(2)} \times \dots \times C_{\sigma(m)})$  for any permutation  $\sigma$  of  $\{1, 2, \dots, m\}$ .

In the following example, we simplify the model of Duffie-Gârleanu-Pedersen [6] by keeping only their binary interacting kernel.

**Example 1** . *Investors in this model have two liquidity states denoted  $h$ , for high, and  $l$ , for low. Moreover, there is an asset of common interest to these investors who either own the asset (denoted by  $o$ ) or don't (denoted by  $n$ ). So  $E = \{(l, n), (l, o), (h, n), (h, o)\}$  describes the state space. The kernel is defined by  $Q_2(\cdot, \cdot; C_1 \times C_2) = 0$  except for*

$$Q_2((h, n), (l, o); C_1 \times C_2) = Q_2((l, o), (h, n); C_2 \times C_1) = \delta_{(h,o)}(C_1)\delta_{(l,n)}(C_2)$$

where  $\delta_{z_0}$  is the Dirac function  $\delta_{z_0}(z) = 1$  iff  $z = z_0$  and  $\delta_{z_0}(z) = 0$  otherwise. The binary kernel implements the trading of the asset whenever a low liquidity investor who owns the asset meets a high liquidity investor who does not yet hold it.

The interactions occur at each jump of a Poisson process with intensity  $\lambda \frac{N}{m}$ . Groups are undistinguishable so each group has a probability of  $\binom{N}{m}^{-1}$  of being involved in a given interaction.

The kernel  $Q$  allows us to describe the macroscopic evolution of the system with an associated system of non-linear differential equations via the evolution of the law of a component. This probability law, denoted  $\mu_t$ , evolves with time and is in fact the solution of the Cauchy problem:

$$\frac{d\mu_t}{dt} = \lambda(\mu_t^{\circ m} - \mu_t); \mu_0 = \mu$$

where

$$\mu^{\circ m}(C) \triangleq \int_{\mathbb{R}^m} \mu(dx_1)\mu(dx_2)\dots\mu(dx_m)Q(x_1, x_2, \dots, x_m; C \times E^{m-1}) \text{ for } C \in \mathcal{E}.$$

The probability law  $\mu^{\circ m}$  is the law of a component after the interaction of  $m$  i.i.d. components with law  $\mu$ . We can think of it as the law at the root of the

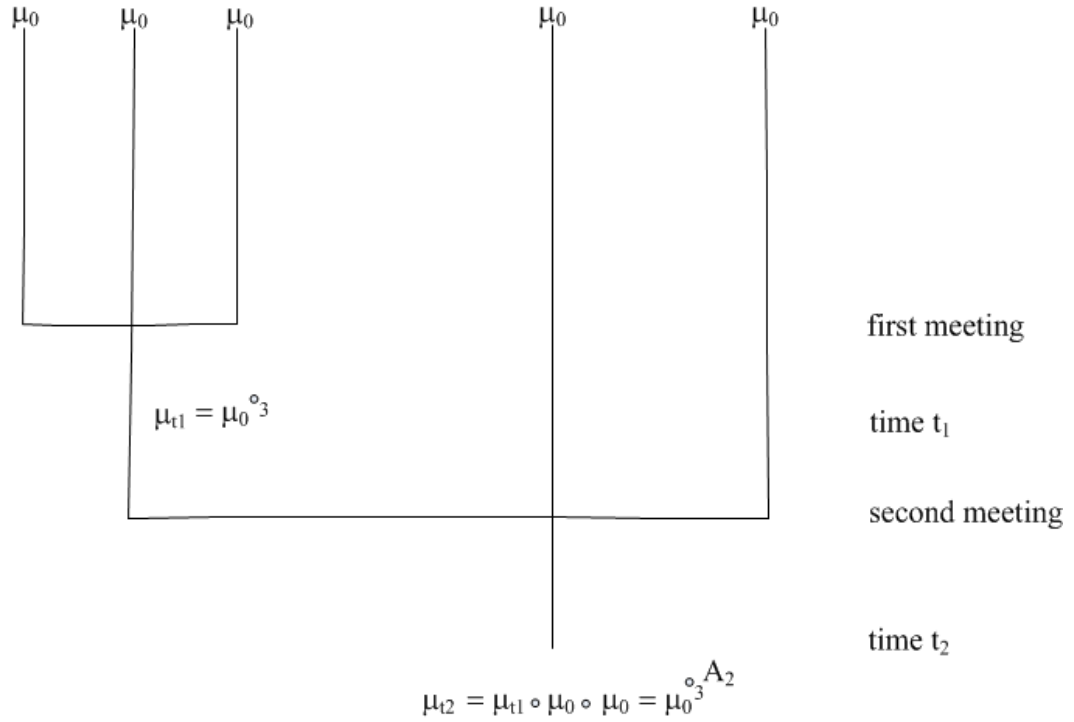


Figure 1: Simple interaction tree with only two meetings and  $m = 3$ .

$m$ -ary tree with only one interaction. We will look at all the trees representing the interaction history of a component up to time  $t$ . So for a tree,  $A$ , with more than one interaction, we divide the tree in  $m$  subtrees at that last interaction and continue recursively up to time 0 to define  $\mu^{\circ_m A}$ . (See Figure 1 for a simple example of an interaction tree.) Let  $\mathbb{A}_n$  be the set of all trees with  $n$  interactions (a.k.a. nodes), each node producing  $m$  branches. If  $A_n \in \mathbb{A}_n$ , then  $\mu^{\circ_m A_n}$  denotes the law obtained by iteration of  $\mu^{\circ_m}$  through the successive nodes of the tree when we place the law  $\mu$  on each leaf of  $A_n$ .

Bélangier and Giroux [1] have shown that the Cauchy problem has a unique solution which can be expressed, by conditioning on the number of interactions up to time  $t$ , and then by the component's history. Such conditionings give us

$$\mu_t = \sum_{n \geq 0} p_n(t) \frac{1}{\#_m(n)} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ_m A_n} \quad (1)$$

where  $\#_m(n) = \prod_{k=1}^{n-1} ((m-1)k+1)$  is the number of trees with  $n$  nodes, taking into account their branching orders; and  $p_n(t) = \frac{\#_m(n)}{(m-1)^n n!} e^{-\lambda t} (1 - e^{-(m-1)\lambda t})^n$  is the probability of having  $n$  branchings up to time  $t$ .

**Remark 2** We call the law  $\mu_t = \sum_{n \geq 0} e^{-\lambda t} (1 - e^{-(m-1)\lambda t})^n \frac{1}{(m-1)^n n!} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ_m A_n}$  an explicit extended Wild sum [15] and note that the convex combination we obtain for the case  $m = 2$  is indeed the Wild sum,  $\mu_t = \sum_{n \geq 0} e^{-\lambda t} (1 - e^{-\lambda t})^n \frac{1}{n!} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ_m A_n}$ , now well-known in the statistical physics of gases since the work of Kac (1956) [8].

### 3.1 Using interaction trees to go from the microscopic to the macroscopic.

In all our cases, we have an underlying market structure which is a Kac walk with interactions involving  $m$  agents. We add exponential times to obtain a marked Poisson process whose marks are horizontal lines linking the agents participating in a given interaction. Bélanger and Giroux [1] describe the limit law of an agent, under an appropriate conditioning, as a countable convex combination on trees which is, as we have shown in Section 3 of that article, the global solution of the associated differential equation on the space of probability laws.

Here we first explain how we came to that convex combination since it serves as a tool to study the other models which follow. It is the tool that enables us, for instance, to state proposition 5. Its proof follows the lines of the proof of the main result in Bélanger and Giroux [1].

We start our study by an analysis of the dynamics of the intrinsic structure of the large set of interacting agents when the number of agents increases. We assume that each interaction involves  $m$  agents,  $m \geq 2$ . More specifically, we consider a set of  $N$  agents whose interactions happen at unexpected times so these interactions' occurrences follow a Poisson process. Since agents are interchangeable, each group has an equal probability of meeting of  $\binom{N}{m}^{-1}$ .

If we suppose the intensity of the meetings to be  $\frac{N}{m}$  then each agent has a meeting rate  $\lambda$  which can be assumed to equal 1 under a time change. We will make this assumption,  $\lambda = 1$ , all throughout Section 3.

For  $N$  fixed and starting at time 0, we assign a vertical position to each agent. The down movement represents the passage of time, see Figure 1 on page 4. Each time a group of agents interacts, we draw a horizontal line between those agents and we draw a vertical line at each agent's position connecting 0 to the horizontal line just drawn, so we see a random graph being formed. When we stop this graph at time  $t$ , we obtain the finite graph of all interactions that have taken place. Moreover, the history up to time  $t$  of a given agent, call it

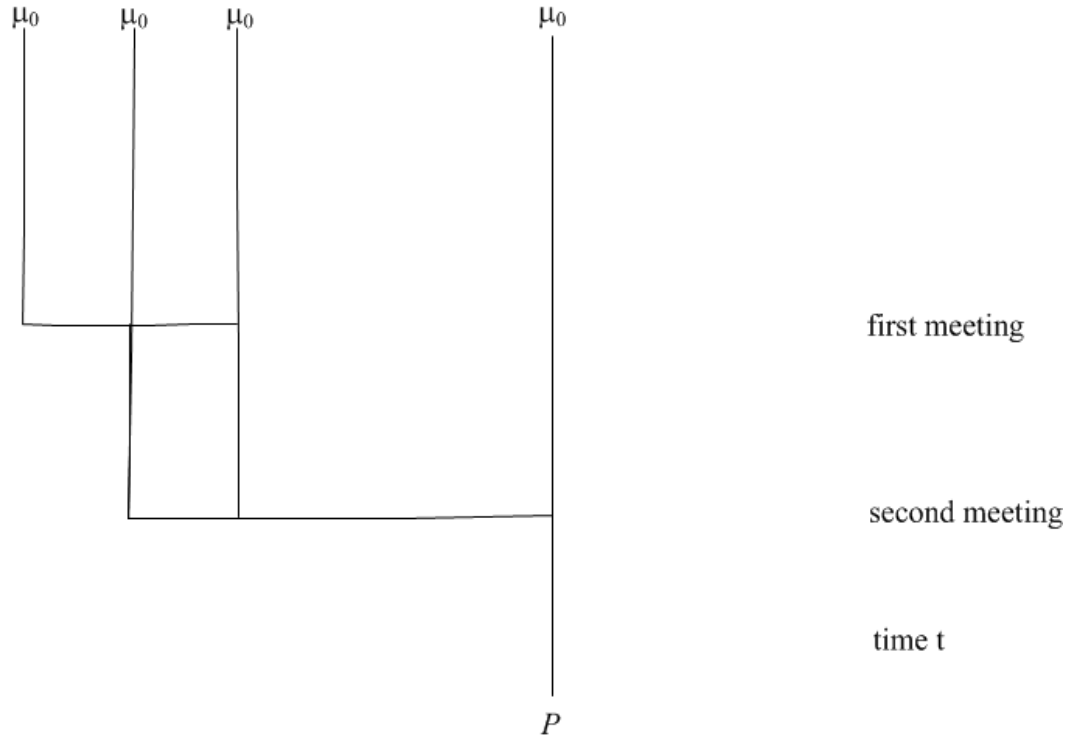


Figure 2: Simple graph with cycle.

$P$ , is described by the random graph connecting all agents who have interacted directly or indirectly with  $P$ .

The number of meetings is random but we can condition on it. The law of the finite graph is reversible since the meeting times are uniform on  $[0, t]$ . We want to show that a random graph representing the history of  $P$  can be replaced by a random tree as the number of agents,  $N$ , grows. If we look at Figure 2, we see that the inclusion in the second meeting of one of the investors having participated in the first one would create a cycle in our graph. As  $N$  grows though, the chance of meeting an investor previously encountered directly or indirectly tends to zero.

To see this, let us consider the graph of  $P$ 's history up to time  $t$ . Starting at time  $t$ , we pursue each one of the encountered vertical lines in  $P$ 's history backward in time until we reach the next horizontal line. If the inclusion of the horizontal line in our graph does not create a cycle (i.e. no pair of investors were involved directly or indirectly in a previous meeting) we include the line, if not we remove it. Proceeding in this fashion up to time 0 we get a tree with  $n$  internal nodes, say, which has the same law as the law of a tree obtained by

a pure-birth process. The tree obtained by a sample history of  $P$ 's interactions is an  $m$ -ary tree.

These trees grow randomly in time: each time a new node appears, corresponding to the occurrence of a meeting of investors at that time. We recall that  $\mathbb{A}_n$  denotes the set of  $m$ -ary ordered trees with  $n$  internal nodes. Then  $\mathbb{A}_n$  constitutes a set of random trees if we assume that every  $m$ -ary tree in  $\mathbb{A}_n$  is equally likely, namely of probability  $\frac{1}{\#\mathbb{A}_n}$ .

The tree starting at  $P$ 's vertical line at time  $t$  with intensity 1 and which at time 0 has intensity  $(m-1)n+1$  and that same number of leaves. Between two branchings of this process a graph representing  $P$ 's meeting history can have a random number of additional horizontal lines following a Poisson law of parameter at most  $\frac{N}{m} \left( \binom{(m-1)n+1}{2} \binom{N}{m}^{-1} \right)$ . We will now bound the expectation of these supplementary horizontal lines by a majorant which tends to 0 as  $N$  increases. Indeed, since the mean number of redundant lines when there are  $n$  branchings up to time  $t$  is at most  $\frac{N}{m} \left( \binom{(m-1)n+1}{2} \binom{N}{m}^{-1} \right)$ , we have that the mean number of redundant horizontal lines is bounded above by

$$\sum_{n \geq 0} \frac{N}{m} \left( \binom{(m-1)n+1}{2} \binom{N}{m}^{-1} \right) p_{N,n}(t),$$

where  $p_{N,n}(t)$  is the probability of having  $n$  branchings up to time  $t$  of the pure birth process with successive branching waiting times following exponential laws of parameter

$$\lambda_{N,n} = \frac{N}{m} ((m-1)n+1) \binom{N - ((m-1)n+1)}{m-1} \binom{N}{m}^{-1}$$

Since

$$\begin{aligned} & \frac{N}{m} ((m-1)n+1) \binom{N - ((m-1)n+1)}{m-1} \binom{N}{m}^{-1} \\ &= \frac{((m-1)n+1) \binom{N - ((m-1)n+1)}{m-1}}{\binom{N-1}{m-1}} \quad (2) \\ &\leq (m-1)n+1 \end{aligned}$$

then  $p_{N,n}(t)$  is stochastically smaller than the law obtained with the intensities  $\lambda_n = (m-1)n+1$ , which in turn are less than the intensities  $\bar{\lambda}_n = m(n+1)$ . Its transition kernel is then obtained by solving Kolmogorov's affine system of equations:



$$\begin{aligned}\frac{d\bar{p}_t(0)}{dt} &= -m\bar{p}_t(0) \\ \frac{d\bar{p}_t(n)}{dt} &= mn\bar{p}_t(n-1) - m(n+1)\bar{p}_t(n) \quad ; n \geq 1.\end{aligned}$$

Thus the latter intensities give us a geometric law  $\bar{p}_t(n) = e^{-mt}(1 - e^{-mt})^n = e^{-m(n+1)t}(e^{mt} - 1)^n$ . Since geometric laws have finite moments of all orders, the mean number of redundant horizontal lines is bounded above by a quantity converging to 0.

For more details on Kolmogorov systems of equations for pure birth processes we refer the reader to Lefebvre [10], for instance.

Thus, after having specified the initial agents' states and their interaction kernels, we can approximate  $P$ 's law using the tree obtained from removing all redundant horizontal lines from its graph. We will use this fact in the next sub-section.

### 3.2 Limit countable convex combination

We will now show that these random trees whose branching intensities depend on  $N$  can be approximated by trees with branching intensities independent of  $N$ . Taking into account that  $P$ 's tree history is random with intensities depending on  $N$ , we could write  $P$ 's law, denoted by  $\mu_t^{*,N}$ , with complex formulae depending on  $N$ . Since our markets have a large number of investors, it is preferable instead to work with the limit of these laws. We note from (2) above that for each  $n$ ,  $\lambda_{N,n} \rightarrow ((m-1)n+1)$  as an increasing sequence in  $N$ .

Let  $p_n(t) (\triangleq p_t(n))$  be the solution of the affine Kolmogorov system of equations:

$$\begin{aligned}\frac{dp_t(0)}{dt} &= -p_t(0) \tag{3} \\ \frac{dp_t(n)}{dt} &= ((m-1)(n-1)+1)p_t(n-1) - ((m-1)n+1)p_t(n) \quad ; n \geq 1.\end{aligned}$$

Recall from Section 1 that  $\mu_t = \sum_{n \geq 0} p_n(t) \frac{1}{\#_m(n)} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ_m A_n}$ .

**Proposition 3** *The sequence of laws  $\mu_t^{*,N}$  converges to  $\mu_t$  as  $N$  increases.*

**Proof.** By Kurtz [9], we have that  $p_{N,n}(t) \rightarrow p_n(t)$  as  $N$  increases. But  $(p_n(t))_{n \geq 0}$  is a probability law, so for  $\epsilon > 0$ , there exists  $n(\epsilon)$  such that

$\sum_{n \geq n(\epsilon)} p_n(t) < \epsilon$ . Now let  $N(\epsilon)$  be such that  $N > N(\epsilon)$  implies that  $|p_{N,n}(t) - p_n(t)| < \frac{\epsilon}{n(\epsilon)}$  for  $0 \leq n \leq n(\epsilon)$ . We then have for  $C \in \mathcal{E}$  and  $N > N(\epsilon)$

$$|\mu_t^{*,N}(C) - \mu_t(C)| \leq \sum_{n=0}^{n(\epsilon)} |p_{N,n}(t) - p_n(t)| + 2\epsilon \leq 3\epsilon$$

since  $\frac{1}{\#_m(n)} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ_m A_n}(C) \leq 1$  and  $(p_{N,n}(t))_{n \geq 0}$  are probability laws. Our claim is proved. ■

**Lemma 4**  $p_t(n) = \frac{\#_m(n)}{(m-1)^n n!} e^{-t} (1 - e^{-(m-1)t})^n$

**Proof.** We need to solve the affine Kolmogorov system of equations (3).

Proceeding by induction we have:

$$\begin{aligned} \frac{dp_t(0)}{dt} &= -e^{-t} \\ \frac{dp_t(n)}{dt} &= ((m-1)(n-1) + 1)e^{-(n(m-1)+1)t} \int_0^t e^{(n(m-1)+1)s} p_s(n-1) ds \end{aligned}$$

To prove the lemma it suffices to note that  $\#_m(n) = \#_m(n-1)((n-1)(m-1)+1)$  and that  $e^{(n(m-1)+1)s} e^{-s} (1 - e^{-(m-1)s})^{n-1} = e^{(m-1)s} (e^{(m-1)s} - 1)^{n-1}$  is the derivative of  $\frac{1}{(m-1)^n} (e^{(m-1)s} - 1)^n$ . ■

And this shows that the limit law of  $P$  is indeed the extended Wild sum which we have shown (in [1]) to be the solution of the ODE associated to the interacting system.

## 4 Explicit formulas for other OTC market models

In many applications, it is more convenient to work with more than one kernel to describe the dynamics of the system. It is the case for instance in the models of Duffie, Gârleanu and Pedersen [6] and their extensions in Bélanger, Giroux and Moisan [2] and in Bélanger, Giroux and Ndouné [3].

In the simplest such model on  $E = \{(l, n), (l, o), (h, n), (h, 0)\}$  we have the binary kernel we described at the beginning of Section 3 and we have the autonomous changes of liquidity of an investor. Let  $\gamma_u$  and  $\gamma_d$  resp. be the intensity of the up movements (resp. down movements) in liquidity. We will first assume that these intensities are equal (we will remove this assumption at the end of the section) and we let  $\gamma = \gamma_u = \gamma_d$ . Then  $q_p(t) = e^{-\gamma t} \frac{(\gamma t)^p}{p!}$  is the probability

of having  $p$  autonomous movements up to time  $t$ . The 1-ary kernel can then be defined by

$$\begin{aligned} Q_1((l, n); C) &= \delta_{(h, n)}(C); Q_1((l, o); C) = \delta_{(h, o)}(C); \\ Q_1((h, n); C) &= \delta_{(l, n)}(C); Q_1((h, o); C) = \delta_{(l, o)}(C); \end{aligned}$$

and

$$\nu^{\circ 1}(C) = \nu(l, n)\delta_{(h, n)}(C) + \nu(l, o)\delta_{(h, o)}(C) + \nu(h, n)\delta_{(l, n)}(C) + \nu(h, o)\delta_{(l, o)}(C)$$

for  $C \subset E$ .

It is possible to modify the 1-ary kernel into a binary kernel with the addition of a “witness” investor who is completely unaffected by the change of liquidity of the other investor. We then symmetrize that kernel and replace the two binary kernels with a convex combination of the two kernels to be left with only one kernel as in the situation we dealt with in the preceding sections. So all the formalism developed so far is still valid. The drawbacks of this approach though are that the symmetrization operation gives us a slightly different dynamics, and more importantly, that we lose the explicit formulas for the extended Wild sums. The objective of this section is to show how we can retain them.

Let  $\mathbb{K}_n^p$  denote the set of all arrangements of  $p$  undistinguishable objects in  $n$  boxes, where a box may contain arbitrarily many objects. Then  $|\mathbb{K}_n^p| = \binom{n+p-1}{n-1}$ . Let  $\mathbb{A}_n$  denote, as before, the set of all random trees with  $n$   $m$ -ary interactions (the investors meetings). If  $A_n \in \mathbb{A}_n$  then  $A_n$  has  $(m-1)n+1$  leaves which in turn gives  $mn+1$  branches.

Let  $p$  denote the number of 1-ary interactions (i.e. the number of autonomous changes of position). For  $\sigma \in \mathbb{K}_{mn+1}^p$ , let  $A_n^\sigma$  denote the tree obtained by placing the 1-ary interactions on each branch of the tree according to the arrangement  $\sigma$ . Let  $\tilde{\mathbb{A}}_{n,p}$  denote the set of trees with  $n$   $m$ -ary interactions and  $p$  1-ary ones. Then  $\rho: \mathbb{K}_{mn+1}^p \times \mathbb{A}_n \rightarrow \tilde{\mathbb{A}}_{n,p}: (\sigma, A_n) \mapsto A_n^\sigma$  defines a bijection. Please see Figure 2 for simple examples of trees in  $\tilde{\mathbb{A}}_{2,1}$ . Moreover, if we call  $\{\sigma^i\}_{i=1}^7$  the 7 configurations of Figure 3, then  $(\nu^{\circ 3 A_2})^{\circ 1} = \sum_{i=1}^7 \nu^{\circ 3 A_2^{\sigma^i}}$ .

The set  $\tilde{\mathbb{A}}_{n,p}$  of  $(m, 1)$ -ary ordered trees with  $n$   $m$ -ary internal nodes and  $p$  1-ary nodes has the cardinality equal to  $\#_m(n) \binom{mn+p}{mn}$ . Then  $\tilde{\mathbb{A}}_{n,p}$  constitutes a set random trees if we assume that every  $(m, 1)$ -ary tree in  $\tilde{\mathbb{A}}_{n,p}$  is equally likely, namely with probability  $\frac{1}{\#_m(n) \binom{mn+p}{mn}}$ .

**Proposition 5** *The probability measure*

$$\nu_t = \sum_{n \geq 0} p_n(t) \frac{1}{\#_m(n)} \sum_{A_n \in \mathbb{A}_n} \left[ \sum_{p=0}^{\infty} \frac{q_p(t)}{\binom{mn+p}{mn}} \sum_{\sigma \in \mathbb{K}_{mn+1}^p} \nu^{\circ m A_n^\sigma} \right]$$

is the solution of the Cauchy problem:

$$\frac{d\nu_t}{dt} = \lambda(\nu_t^{\circ m} - \nu_t) + \gamma(\nu_t^{\circ 1} - \nu_t); \nu_0 = \nu.$$

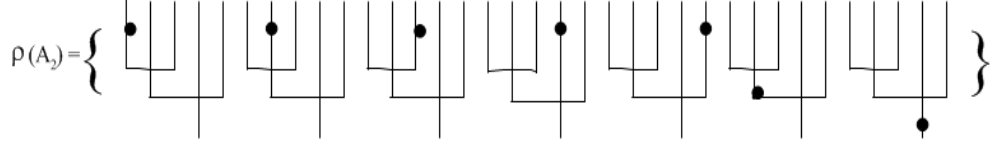


Figure 3: Interaction trees for  $m=3$ ,  $A_2$  and  $p=1$ .

**Proof.**

Let  $\mu_t$  be the solution (1) above. Since  $|\nu_t| \leq |\mu_t|$  and  $\mu_t$  is uniformly summable, the convex sum  $\nu_t$  can be differentiated term by term to obtain:

$$\begin{aligned} \frac{d\nu_t}{dt} = & -\lambda\nu_t + \\ & \lambda e^{-m\lambda t} \sum_{n \geq 1} (1 - e^{-(m-1)\lambda t})^{n-1} \frac{1}{(m-1)^{n-1} (n-1)!} \sum_{A_n \in \mathbb{A}_n} \left[ \sum_{p=0}^{\infty} \frac{q_p(t)}{\binom{mn+p}{mn}} \sum_{\sigma \in \mathbb{K}_{mn+1}^p} \nu^{\circ_m A_n^\sigma} \right] \\ & - \gamma \nu_t + \sum_{n \geq 0} p_n(t) \frac{1}{\#_m(n)} \sum_{A_n \in \mathbb{A}_n} \left[ \sum_{p=1}^{\infty} \frac{e^{-\gamma t} (\gamma t)^{p-1}}{(p-1)! \binom{mn+p}{mn}} \sum_{\sigma \in \mathbb{K}_{mn+1}^p} \nu^{\circ_m A_n^\sigma} \right] \end{aligned}$$

Now using the combinatorial results of the proof of theorem 1 in Bélanger and Giroux [1], we have that  $\nu_t^{\circ_m}$  is equal to the expression

$$\lambda e^{-m\lambda t} \sum_{n \geq 1} (1 - e^{-(m-1)\lambda t})^{n-1} \frac{1}{(m-1)^{n-1} (n-1)!} \sum_{A_n \in \mathbb{A}_n} \left[ \sum_{p=0}^{\infty} \frac{q_p(t)}{\binom{mn+p}{mn}} \sum_{\sigma \in \mathbb{K}_{mn+1}^p} \nu^{\circ_m A_n^\sigma} \right].$$

Otherwise, the expression

$$\sum_{n \geq 0} p_n(t) \frac{1}{\#_m(n)} \sum_{A_n \in \mathbb{A}_n} \left[ \sum_{p=1}^{\infty} \frac{e^{-\gamma t} (\gamma t)^{p-1}}{(p-1)! \binom{mn+p}{mn}} \sum_{\sigma \in \mathbb{K}_{mn+1}^p} \nu^{\circ_m A_n^\sigma} \right]$$

is equal to the quantity

$$\sum_{n \geq 0} p_n(t) \frac{1}{\#_m(n)} \sum_{A_n \in \mathbb{A}_n} \left[ \sum_{p=0}^{\infty} \frac{e^{-\gamma t} (\gamma t)^p}{p! \binom{mn+p+1}{mn}} \sum_{\sigma \in \mathbb{K}_{mn+1}^{p+1}} \nu^{\circ_m A_n^\sigma} \right].$$

But  $\sum_{\sigma \in \mathbb{K}_{mn+1}^{p+1}} \nu^{\circ_m A_n^\sigma} = \sum_{\sigma' \in \mathbb{K}_{mn+1}^p} (\nu^{\circ_m A_n^{\sigma'}})^{\circ_1}$ . And then

$$\sum_{n \geq 0} p_n(t) \frac{1}{\#_m(n)} \sum_{A_n \in \mathbb{A}_n} \left[ \sum_{p=0}^{\infty} \frac{e^{-\gamma t} (\gamma t)^p}{p! \binom{mn+p+1}{mn}} \sum_{\sigma' \in \mathbb{K}_{mn+1}^p} (\nu^{\circ_m A_n^{\sigma'}})^{\circ_1} \right] = \nu_t^{\circ_1}.$$

Hence we get that  $\frac{d\nu_t}{dt}$  has the desired form and the proof of the proposition is now complete.

■

We note that if  $p = 0$ , that is, if no investor changes its liquidity position, the above solution does indeed become the solution (1).

**Remark 6** *In the specific context of the DGP model we have a binary kernel which simplifies the first part of the formula. But without the assumption  $\gamma_u = \gamma_d$ , we have to consider the up movements and the down movements separately, and this makes for a more complicated second part of the formula. Let  $\gamma = \gamma_u + \gamma_d$ , the solution becomes*

$$\nu_t = \sum_{n \geq 0} \frac{e^{-t}(1 - e^{-t})}{n!} \sum_{A_n \in \mathbb{A}_n} \left[ \sum_{p=0}^{\infty} \frac{e^{-\gamma t}}{p! \binom{2n+p+1}{2n}} \left[ \sum_{k=0}^p \frac{(\gamma_u t)^k (\gamma_d t)^{p-k}}{\binom{p}{k}} \sum_{\substack{\sigma_u \in \mathbb{K}_{2n+1}^k \\ \sigma_d \in \mathbb{K}_{2n+1}^{p-k}}} \nu^{\circ_2 A_n^{\sigma_u \cup \sigma_d}} \right] \right]$$

where  $\sigma_u$  (resp  $\sigma_d$ ) denotes the arrangements of up movements (resp. down movements) on the branches of the tree and  $\sigma_u \cup \sigma_d$  is the arrangement obtained from both arrangements of up and down movements.

**Remark 7** *We can obtain similar explicit formulas for OTC models where the interactions involve  $m > 2$  investors. In the information percolation model of Duffie-Malamud-Manso[7], for instance, the state space,  $E = \mathbb{N}$  represents the potential levels of information acquired by an investor through meetings with other investors. The  $m$ -ary interaction is the perfect sharing of information which means that each investor in the meeting comes out with the sum of the information levels of all participating investors. The unary kernel is a regression force which replaces an investor of level  $n$  say, by an investor with level  $\pi(n)$  sampled from a given distribution  $\pi$  on  $\mathbb{N}$ .*

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