Interacting Sets of Investors with Exits

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Avant-propos pour les cahiers de recherche du GReFA

Dans un cahier précédent, intitulé "Extended Kac Walks and an Application in Econophysics", nous avons généralisé l'étude de la dynamique des marchés OTC comme grands ensembles interactifs où les interactions se produisent à m agents avec nombre d'agents $m \geq 2$ quelconque. Poursuivant cette direction, nous généralisons, dans ce cahier, les résultats d'existence et d'unicité d'une loi stable pour un modèle d'échange d'information binaire de Duffie et al (Duffie,D., Malamud, S. and Manso, G., "Information percolation with equilibrium search dynamics", *Econometrica*, vol. 77, no. 5, 1513-1574, (2009).) au cas d'échange multipartite. Nous montrons aussi l'utilité de notre résultat sur l'existence d'une solution globale du problème de Cauchy pour une grande classe d'équations différentielles non-linéaires, en obtenant la stabilité exponentielle de cas particuliers de systèmes d'un nombre fini d'équations.

Abstract

We obtain the global solution of the Cauchy problem of a large class of non-linear differential equations. This enables us to deduce the exponential stability of particular systems with a finite number of equations. We moreover extend some results on a binary information exchange model to the case where the exchanges involve m agents, for $m \geq 2$.

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1 Introduction

In [2] Duffie-Malamud-Manso (2009), the authors have obtained interesting results for binary information exchange models in large interacting sets where agents have an autonomous movement which makes them quit the market in order to be replaced by a new agent whose degree of precision follows a given probability law. The evolution of the agent's law in such a set can be approximated by the solution of a quadratic system which, in the binary case with perfect information transmission, can be written

$$\frac{d\mu_t(n)}{dt} = \eta(\pi(n) - \mu_t(n)) + \sum_{l=1}^{n-1} h(l, n-l)\mu_t(l)\mu_t(n-l) - \sum_{l\geq 1} h(n,l)\mu_t(n)\mu_t(l)$$

where π is a given probability law.

It is important to note that the last term is linear in $\mu_t(l)$ since it is the use of that fact which enables [2] Duffie *et al* to obtain the stability of such systems when $h(n,m) = c(n)c(m)$, for a positive function c which is constant above a certain n, high enough, and such that η is bigger than c times this constant. There are many ways to generalize this system of equations. The meetings can involve more than 2 agents, like in private auctions, and the information transmission might be imperfect. The transmission kernel which can be written in the case of perfect transmission with m agents

$$\begin{aligned} & Q_{pf}(n_1, n_2, ..., n_m; l_1, l_2, ..., l_m) \\ = \quad & \delta_{n_1+n_2+...+n_m}(l_1)\delta_{n_1+n_2+...+n_m}(l_2)\cdots\delta_{n_1+n_2+...+n_m}(l_m) \end{aligned}$$

is then replaced by an arbitrary symmetric probability kernel K.

We will show, in section 2, the existence of a global solution of the following general systems:

$$\frac{d\mu_t(n)}{dt} = \eta(\pi(n)-\mu_t(n))+\sum_{n_1,...,n_m} \mu_t(n_1)\cdots\mu_t(n_m)h(n_1,...,n_m)(\overline{K}(n_1,...,n_m;n)-1_n)$$

where $\overline{K}(n_1,...,n_m;n) = \sum_{i_2,...,i_m} K(n_1,...,n_m;n,i_2,...,i_m)$ when the intensity function h is bounded. In the particular case of perfect exchange for m agents the system can be written

$$\begin{aligned}\frac{d\mu_t(n)}{dt} &= \eta(\pi(n)-\mu_t(n)) + \sum_{n_1+...+n_m=n} \mu_t(n_1)\cdots\mu_t(n_m)h(n_1,...,n_m) \\ &\quad - \sum_{n_2,...,n_m} \mu_t(n)\cdots\mu_t(n_m)h(n,n_2,...,n_m)\end{aligned}$$

Our result allows us to state the existence of a global solution of a large number of non-linear differential systems of a finite number of equations like for example

$$\begin{aligned}\frac{dx}{dt} &= 1-x-x^m \\ \frac{dy}{dt} &= -y+x^m\end{aligned}$$

since this system can be obtained in our framework when $\eta = 1, \pi(1) = 1$, $h(1,1,...,1) = 1$ and h is equal to 0 elsewhere.

Note that for $m = 2$, the solution of $\frac{dx}{dt} = 1-x-x^2$ can be obtained by separation of variables, see [1] Bahk-Dyakevich-Johnson (2008) for a general study of the Riccati equation.

We could think of using this technique to solve the equation in general but the analysis of [1] would involve a factorization of the polynomial $1-x-x^m$ for a general m which seems difficult. On the other hand, the existence of a solution for all times t cannot be obtained from the classical theory of ODE's which is primarily concerned with the existence of local solutions.

There are obviously many other examples involving symmetric functions h. For example, if we let $a \geq 0, b \geq 0, h(1,1,...,1) = a, h(m,1,1,...,1) = h(1,m,1,...,1) = ... = h(1,1,...,1,m) = b$ with all other values being 0 we obtain the following system:

$$\begin{aligned}\frac{dx}{dt} &= 1 - x - ax^m - (m-1)bx^{m-1}y \\ \frac{dy}{dt} &= -y + ax^m - bx^{m-1}y \\ \frac{dz}{dt} &= -z + mbx^{m-1}y\end{aligned}$$

(which reduces to the previous sytem when $a = 1$ and $b = 0$). We will study this system in section 3.

But we will first demonstrate, in section 2, the existence of the global solution of the Cauchy problem under the only hypothesis that h is positive and bounded. In section 3, we will show, with a few examples of systems with a finite number of equations, how we can obtain a general solution. We will moreover obtain stable points for these systems and we will show their asymptotic stability at an exponential rate. In section 4, we will return to the particular case of [2] Duffie *et al* with meetings involving m agents and we will demonstrate the existence of invariant laws.

2 Existence of global solutions

The general systems we will consider in this section can be written as follows:

$$\frac{d\mu_t(n)}{dt} = \eta(\pi(n) - \mu_t(n)) + \sum_{n_1,...,n_m} \mu_t(n_1) \cdots \mu_t(n_m) h(n_1,...,n_m)(\overline{K}(n_1,...,n_m;n) - 1_n)$$

where $\overline{K}(n_1, ..., n_m; n) = \sum_{i_2,...,i_m} K(n_1, ..., n_m; n, i_2, ..., i_m)$ when the intensity function h is bounded. Let $h^* = \sup h(n_1, ..., n_m)$. Then

$$\frac{d\mu_t(n)}{dt} = \eta(\pi(n) - \mu_t(n)) +$$

$$+h^* \left\{ \sum_{n_1,...,n_m} \mu_t(n_1) \cdots \mu_t(n_m) (\frac{h(n_1, ..., n_m)}{h^*} (\overline{K}(n_1, ..., n_m; n) - 1_n) \right.$$
$$\left. + \left(1 - \frac{h(n_1, ..., n_m)}{h^*}\right) (\overline{I}(n_1, ..., n_m; n) - 1_n) \right\}$$

$$\text{where } \overline{I}(n_1, ..., n_m; n) = \begin{cases} 1 \text{ if } n_1 = n \\ 0 \text{ otherwise} \end{cases} .$$

Let

$$\begin{aligned} \overline{J}(n_1, ..., n_m; n) &= \frac{h(n_1, ..., n_m)}{h^*} (\overline{K}(n_1, ..., n_m; n) - 1_n) \\ &\quad + \left(1 - \frac{h(n_1, ..., n_m)}{h^*}\right) (\overline{I}(n_1, ..., n_m; n) - 1_n)) \end{aligned}$$

These systems can be rewritten:

$$\begin{aligned} \frac{d\mu_t(n)}{dt} &= (\eta + h^*) \frac{\eta}{\eta + h^*} (\pi(n) - \mu_t(n)) \\ &\quad + (\eta + h^*) \frac{h^*}{\eta + h^*} \left\{ \sum_{n_1,...,n_m} \mu_t(n_1) \cdots \mu_t(n_m) \overline{J}(n_1, ..., n_m; n) \right\} \end{aligned}$$

If we define the kernel $K_\pi(n_1, ..., n_m; l_1, ..., l_m) = \sum \frac{1}{m} \pi(l_i)$ and if we do a time change (so that $\eta + h^* = 1$) then the interaction equation can be written

$$\frac{d\mu_t(n)}{dt} = \sum_{n_1,...,n_m} \mu_t(n_1) \cdots \mu_t(n_m) (Q(n_1, ..., n_m) - 1_n)$$

where Q is a symmetric probability kernel. Such an equation, in turn, is a particular case of the following equation

$$\frac{d\mu_t}{dt} = \mu_t^{\circ_m} - \mu_t$$

where m is an integer greater or equal to 2; $(E, \mathcal{E})$ is a measurable space; Q is a symmetric probability kernel on

$(E^m, \mathcal{E}^{\otimes m})$; $\overline{Q}(x_1, x_2, ..., x_m; C) = Q(x_1, x_2, ..., x_m; C \times E^{m-1})$ for $C \in \mathcal{E}$; and $\mu^{\circ_m}(C) \triangleq \int\limits_{\mathbb{R}^m} \mu(dx_1)\mu(dx_2)\cdots\mu(dx_m)\overline{Q}(x_1, x_2, ..., x_m; C)$. Let $\mathbb{A}_n$ be the set of all trees with n nodes, each node producing m branches. If $A_n \in \mathbb{A}_n$, let $\mu^{\circ_m A_n}$ denote the law obtained by iteration of μ°_m} when we place the law μ on each leaf of A_n.

Theorem 1 *The convex combination,*

$$\mu_t = \sum_{n\geq 0} p_n(t) \frac{1}{\#_m(n)} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ_m A_n},$$

is a solution of the Cauchy problem $\frac{d\mu_t}{dt} = \mu_t^{\circ_m} - \mu_t; \mu_0 = \mu$. *Where* $\#_m(n) = \prod_{k=1}^{n-1}((m-1)k+1)$ *is the number of trees with* n *nodes, taking into account their branching orders and* $p_t(n) = \frac{\#_m(n)}{(m-1)^n n!} e^{-t}(1-e^{-(m-1)t})^n$ *is the probability of having* n *branchings up to time* t.

Proof. We differentiate μ_t term by term to obtain:

$$-\mu_t + e^{-mt} \sum_{n\geq 1} n(1-e^{-(m-1)t})^{n-1} \frac{1}{(m-1)^{n-1}(n-1)!} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ_m A_n}$$

Thus we need to show that:

$$\mu_t^{\circ_m}(C) = e^{-mt} \sum_{n\geq 0} n(1-e^{-(m-1)t})^{n-1} \frac{1}{(m-1)^{n-1} n!} \sum_{A_{n+1} \in \mathbb{A}_{n+1}} \mu^{\circ_m A_{n+1}}(C) \qquad (*)$$

But the LHS is equal by definition to

$$\int\limits_{\mathbb{R}^m} \left(\sum_{i_1 \geq 0} e^{-t}(1-e^{-(m-1)t})^{i_1} \frac{1}{(m-1)^{i_1} i_1!} \sum_{A_{i_1} \in \mathbb{A}_{i_1}} \mu^{\circ_m A_{i_1}}(dx_1) \right) ...$$

$$... \left(\sum_{i_m \geq 0} e^{-t}(1-e^{-(m-1)t})^{i_m} \frac{1}{(m-1)^{i_m} i_m!} \sum_{A_{i_m} \in \mathbb{A}_{i_m}} \mu^{\circ_m A_{i_m}}(dx_m) \right) ...$$

$$...Q(x_1, ..., x_m; C \times E^{m-1})$$

which is equal to

$$\int\limits_{\mathbb{R}^m} e^{-mt} \left\{ \sum_{n \geq 0} (1-e^{-(m-1)t})^n \sum_{i_1+...+i_m=n} \frac{1}{(m-1)^n i_1!...i_m!} ... \right.$$

$$\left. \left(\sum_{A_{i_1} \in \mathbb{A}_{i_1}} \mu^{\circ_m A_{i_1}}(dx_1) \right) \cdots \left(\sum_{A_{i_m} \in \mathbb{A}_{i_m}} \mu^{\circ_m A_{i_m}}(dx_m) \right) \right\} Q(x_1, ..., x_m; C \times E^{m-1})$$

which in turn is equal to

$$\int\limits_{\mathbb{R}^m} e^{-mt} \left(\sum_{n \geq 0} (1-e^{-(m-1)t})^n \frac{1}{(m-1)^n n!} F(i_1, ..., i_m, n, \mu, A_{i_1}, ..., A_{i_m}, Q, C) \right)$$

where

$$F(i_1, ..., i_m, n, \mu, A_{i_1}, ..., A_{i_m}, Q, C) = ...$$

$$\sum_{i_1+...+i_m=n} \binom{n}{i_1} \binom{n-i_1}{i_2} \cdots \binom{i_{m-1}+i_m}{i_{m-1}} \left(\sum_{A_{i_1} \in \mathbb{A}_{i_1}} \mu^{\circ_m A_{i_1}}(dx_1) \right) \cdots$$

$$\cdots \left(\sum_{A_{i_m} \in \mathbb{A}_{i_m}} \mu^{\circ_m A_{i_m}}(dx_m) \right) Q(x_1, ..., x_m; C \times E^{m-1})$$

And this last expression is a decomposition of the trees A_{n+1} appearing in the RHS of $(*)$ in m subtrees after the first node (taking the branching order into account). The two expressions are therefore equal and this proves the result. ■

This shows that the particular systems we study here all have a global solution. We will show in the next section that it is advantageous to know that fact when we analyze the stability of systems of differential equations.

3 Systems with a finite number of equations

3.1 Existence of an invariant law

We will show directly that the following system has an invariant solution:

$$\begin{aligned}\frac{dx}{dt} &= \eta(\pi(1) - x) - ax^m - (m-1)bx^{m-1}y \\ \frac{dy}{dt} &= \eta(\pi(m) - y) + ax^m - bx^{m-1}y \\ \frac{dz}{dt} &= \eta(\pi(2m-1) - z) + mbx^{m-1}y\end{aligned}$$

with $\pi(1) + \pi(m) + \pi(2m-1) = 1$. Evidently, such a solution must satisfy the following identities:

$$\begin{aligned}0 &= \eta(\pi(1) - x) - ax^m - (m-1)bx^{m-1}y \\ 0 &= \eta(\pi(m) - y) + ax^m - bx^{m-1}y\end{aligned}$$

Inspired by [2], we proceed as follows. For $C \geq 0$, let

$$x(C) = \frac{\eta\pi(1)}{\eta + C} \tag{1}$$

$$y(C) = \frac{\eta\pi(m) + ax(C)^m}{\eta + bx(C)^{m-1}} \tag{2}$$

Note that $C - \big(ax(C)^{m-1} + (m-1)bx(C)^{m-2}y(C)\big)$ is negative at $C = 0$ and since $x(C)$ is a decreasing function of C the expression is positive for C

high enough. Hence there is a $C^* \geq 0$ such that

$$C^* x(C^*) = \big(ax(C^*)^m + (m-1)bx(C^*)^{m-1}y(C^*)\big).$$

Substituting this value in (1) and (2) gives us that $(x(C^*), y(C^*))$ satisfies the identities. If we furthermore let

$$z(C^*) = \frac{\eta\pi(2m-1) + mbx(C^*)^{m-1}y(C^*)}{\eta}$$

then $(x(C^*), y(C^*), z(C^*))$ is an invariant solution and

$$\begin{aligned}\eta(x(C^*) + y(C^*) + z(C^*)) &= \eta(\pi(1) + \pi(m) + \pi(2m-1)) \\ &\quad -C^* x(C^*) + \big(ax(C^*)^m + (m-1)bx(C^*)^{m-1}y(C^*)\big) \\ &= \eta(\pi(1) + \pi(m) + \pi(2m-1))\end{aligned}$$

which shows that the solution is a probability law.

3.2 Exponential asymptotic stability

It is well-known that a system of differential equations is exponentially stable when the invariant solution of the vector

$$(\eta(\pi(1) - x) - ax^m - (m-1)bx^{m-1}y, \eta(\pi(m) - y) + ax^m - bx^{m-1}y)$$

has a Jacobian matrix whose eigenvalues have strictly negative real parts. We will show that this is indeed the case for all $x, y \in [0, 1]$. If $\begin{bmatrix} u & z \\ w & v \end{bmatrix}$ denote the Jacobian matrix of the vector, then : $u = -\eta - max^{m-1} - (m-1)^2 bx^{m-2}y$; $z = -(m-1)bx^{m-1}$; $w = max^{m-1} - (m-1)bx^{m-2}y$; $v = -\eta - bx^{m-1}$. And direct computations show that its eigenvalues have the form $\frac{\rho \pm \sqrt{\rho^2 - \theta}}{2}$ where θ

is strictly positive and ρ is strictly negative.

4 Extension of the DMM model to the case of interactions with m agents.

Recall that the evolution equation, denoted (*), in the model with perfect information transmission but when interactions involve m agents, is written:

$$\begin{aligned}\frac{d\mu_t(n)}{dt} &= \eta(\pi(n)-\mu_t(n)) \\ &+ \sum_{n_1+...+n_m=n} \mu_t(n_1)\cdots\mu_t(n_m)h(n_1,...,n_m) \\ &- \sum_{n_2,...,n_m} \mu_t(n)\cdots\mu_t(n_m)h(n,n_2,...,n_m)\end{aligned}$$

We will obtain an invariant law in the case where $h(n_1,...,n_m) = c(n_1)\cdots c(n_m)$ with c bounded. Let $\overline{d} = \max\{c(n)\}$. In order to avoid technicalities we will suppose that the support of π is $\{1+k(m-1)\}_{k\geq 0}$.

Proposition 2 *For $d \geq 0$, let*

$$\mu_1(d) = \frac{\eta\pi(1)}{\eta + c(1)d^{m-1}}$$

and define recursively

$$\mu_{1+k(m-1)}(d) = \frac{\eta\pi(1+k(m-1)) + \sum_{n_1+...+n_m=1+k(m-1)} c(n_1)\cdots c(n_m)\mu_{n_1}(d)\cdots\mu_{n_m}(d)}{\eta + c(1+k(m-1))d^{m-1}}$$

The equation

$$d = \sum_{1+k(m-1)\geq 1} c(1+k(m-1))\mu_{1+k(m-1)}(d)$$

admits a unique solution d^**. Moreover the law* $\mu(d^*)$ *is the unique invariant law of the system (*).*

Proof. In order to simplify the notations, let $\overline{c}_k = c(1+k(m-1)); \overline{\pi}_k = \pi(1+k(m-1))$ and $\overline{\mu}_k(d) = \mu_{1+k(m-1)}(d)$. Since $\eta\overline{\mu}_0(d) = \eta\overline{\pi}_0 - \overline{c}_0 d^{m-1}\overline{\mu}_0(d) \leq \eta\overline{\pi}_0$, then $0 \leq \overline{\mu}_0(d) \leq 1$. Similarly,

$$\eta(\overline{\mu}_0(d) + \overline{\mu}_1(d)) = \eta(\overline{\pi}_0 + \overline{\pi}_1) + \overline{c}_0^m\overline{\mu}_0(d)^m - d^{m-1}(\overline{c}_0\overline{\mu}_0(d) + \overline{c}_1\overline{\mu}_1(d))$$

If $d \geq \overline{d}$ then

$$\eta(\overline{\mu}_0(d) + \overline{\mu}_1(d)) \leq \eta + \overline{d}^{m-1}\overline{c}_0\overline{\mu}_0(d) - d^{m-1}\overline{c}_0\overline{\mu}_0(d) \leq \eta$$

Note that $1 + k_1(m-1) + 1 + k_2(m-1) + ... + 1 + k_m(m-1) = 1 + \left(\sum_{j=1}^{m} k_i + 1\right)(m-1)$. Suppose now that $0 \leq \sum_{j=0}^{k} \overline{\mu}_j(d) \leq 1$. Then

$$\begin{aligned}
\eta\left(\sum_{j=0}^{k+1}\overline{\mu}_j(d)\right) &= \eta\left(\sum_{j=0}^{k+1}\overline{\pi}_j(d)\right) \\
&\quad + \sum_{j=0}^{k+1}\sum_{k_1+\ldots+k_m+1=j} \overline{c}_{k_1}\cdots\overline{c}_{k_m}\overline{\mu}_{k_1}(d)\cdots\overline{\mu}_{k_m}(d) \\
&\quad - \sum_{j=0}^{k+1}\overline{c}_j\overline{\mu}_j(d)d^{m-1}
\end{aligned}$$

But

$$
\begin{aligned}
& \sum_{j=0}^{k} \sum_{k_1+\ldots+k_m=j} \overline{c}_{k_1} \cdots \overline{c}_{k_m} \overline{\mu}_{k_1}(d) \cdots \overline{\mu}_{k_m}(d) \\
\leq & \sum_{k_1,\ldots,k_{m-1}=0}^{k} \overline{c}_{k_1} \cdots \overline{c}_{k_{m-1}} \overline{\mu}_{k_1}(d) \cdots \overline{\mu}_{k_{m-1}}(d) \left(\sum_{j=0}^{k} \overline{c}_j \overline{\mu}_j(d) \right) \\
\leq & \sum_{k_1,\ldots,k_{m-1}=0}^{k} \overline{c}_{k_1} \cdots \overline{c}_{k_{m-1}} \overline{\mu}_{k_1}(d) \cdots \overline{\mu}_{k_{m-1}}(d) d \\
\leq & \sum_{k_1,\ldots,k_{m-2}=0}^{k} \overline{c}_{k_1} \cdots \overline{c}_{k_{m-1}} \overline{\mu}_{k_1}(d) \cdots \overline{\mu}_{k_{m-2}}(d) d^2 \ldots\ldots \\
\leq & \sum_{k_1=0}^{k} \overline{c}_{k_1} \overline{\mu}_{k_1}(d) d^{m-1}
\end{aligned}
$$

Hence $0 \leq \sum_{j=0}^{\infty} \overline{\mu}_j(d) \leq 1$. Define $f(d) = \sum_{k=0}^{\infty} \overline{c}_k \overline{\mu}_k(d)$. Since $f(d) \leq d$ for $d \geq \overline{d}$ and $f(0) \geq 0$ there exists d^* such that $f(d^*) = d^*$. Now

$$
\begin{aligned}
\frac{d\overline{\mu}_k(d^*)}{dt} &= \eta(\overline{\pi}_k - \overline{\mu}_k(d^*)) \\
&+ \sum_{k_1+\ldots+k_m=k-1} \overline{\mu}_{k_1}(d^*) \cdots \overline{\mu}_{k_m}(d^*) \overline{c}_{k_1} \cdots \overline{c}_{k_m} \\
&- \sum_{k_2,\ldots,k_m} \overline{\mu}_k(d^*) \overline{\mu}_{k_2}(d^*) \cdots \overline{\mu}_{k_m}(d^*) \overline{c}_k \overline{c}_{k_2} \cdots \overline{c}_{k_m}
\end{aligned}
$$

By definition of d^*

$$
\sum_{k_2,\ldots,k_m} \overline{\mu}_k(d^*) \overline{\mu}_{k_2}(d^*) \cdots \overline{\mu}_{k_m}(d^*) \overline{c}_k \overline{c}_{k_2} \cdots \overline{c}_{k_m} = \overline{c}_k \overline{\mu}_k(d^*) \left(d^*\right)^{m-1}
$$

And by the definition of $\overline{\mu}_k$, we get that $\frac{d\overline{\mu}_k(d^*)}{dt} = 0$. Note moreover that

$$
\begin{aligned}
\sum_{k=0}^{\infty} \sum_{k_2,\ldots,k_m} \overline{\mu}_k(d^*) \overline{\mu}_{k_2}(d^*) \cdots \overline{\mu}_{k_m}(d^*) \overline{c}_k \overline{c}_{k_2} \cdots \overline{c}_{k_m} &= \sum_{k=0}^{\infty} \overline{c}_k \overline{\mu}_k(d^*) \left(d^*\right)^{m-1} \\
&= \left(d^*\right)^m
\end{aligned}
$$

and similarly

$$\sum_{k=0}^{\infty} \sum_{k_1+\ldots+k_m=k-1} \overline{\mu}_{k_1}(d^*) \cdots \overline{\mu}_{k_m}(d^*) \overline{c}_{k_1} \cdots \overline{c}_{k_m} = (d^*)^m$$

So $\{\overline{\mu}_k\}$ is a probability law and its uniqueness follows from [2] pp. 32-36. ■

5 Annex

There is an alternative approach, to the one presented in the first two sections, to obtain the global solution of the evolution equation of a large interacting set. It can be briefly described as follows.

Let the process $\left(Z_i^N(t), ..., Z_N^N(t) : 0 \leq t \leq T\right)_{N\geq 2}$ be defined by an interacting set of $N \geq m$ particles whose interactions follow a Poisson process with intensity $\frac{N}{m}$. Each group of m particles has a probability of $\binom{N}{m}^{-1}$ of being involved in the interaction. The result of the interaction is given by a symmetric probability kernel, Q on $(E^m, \mathcal{E}^{\otimes m})$ (the m-fold product of a measurable space $(E, \mathcal{E})$). This description leads to the homogeneous Markov process $\left(Z_i^N(t), ..., Z_N^N(t) : 0 \leq t \leq T\right)_{N\geq 2}$.

The global solution of the evolution equation is then obtained as the limit of the sequence, $\left(\mu_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{Z_i^N(t)} : 0 \leq t \leq T\right)_{N\geq 2}$, of the empirical measures process along well-chosen test functions. More specifically, we take the limit of the sequences $\left(< \mu_t^N, \varphi >= \frac{1}{N} \sum_{i=1}^{N} \varphi(Z_i^N(t)) : 0 \leq t \leq T\right)_{N\geq 2}$ when φ belongs to a countable dense set of bounded and continuous functions. As usual, the problem is solved in two steps: first by establishing the relative compactness of the sequence and second by showing that the set has a unique limit point. In order to show the relative compactness, it suffices to show that

$$\mathbb{E}\left[\left(< \mu_t^N, \varphi > - < \mu_r^N, \varphi >\right)^2 \left(< \mu_r^N, \varphi > - < \mu_s^N, \varphi >\right)^2\right] \leq C(t-s)^2$$

for $0 \leq s \leq r \leq t \leq T$. This is obtained using the jumps compensators and square field operator of the process $\left(Z_i^N(t), ..., Z_N^N(t) : 0 \leq t \leq T\right)_{N \geq 2}$. Once this is obtained, there only remains to show that the limit points are the Dirac delta mass on the solution of the non-linear equation. See [3] Ferland-Giroux (2008) for the details and proofs.

6 References

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